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Stability problem for the Gołąb–Schinzel type functional equations

Jacek Chudziak

Department of Mathematics, University of Rzeszów, Rejtana 16 A, 35-310 Rzeszów, Poland

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Abstract

Let X be a vector space over a field K of real or complex numbers, $n \in \mathbb{N}$ and $\lambda \in K \setminus \{0\}$. We study the stability problem for the Gołąb–Schinzel type functional equations

$$f\big(x + f(x)^n y\big) = \lambda f(x) f(y)$$

in the class of functions $f: X \to K$ such that the set $\{x \in X\colon\ f(x) \neq 0\}$ has an algebraically interior point.

Keywords: Gołąb–Schinzel equation; Stability; Algebraically interior point; Strongly invariant set

1. Introduction

Let X be a vector space over a field K of real or complex numbers. The Gołąb–Schinzel equation

$$f\big(x + f(x)y\big) = f(x)f(y) \quad \text{for } x, y \in X, \tag{1}$$

and its generalizations

$$f\big(x + f(x)^n y\big) = \lambda f(x) f(y) \quad \text{for } x, y \in X, \tag{2}$$

where $n \in \mathbb{N}$, $\lambda \in K \setminus \{0\}$ are fixed and $f : X \to K$ is an unknown function, play a prominent role in the theory of functional equations. Equations of that type are intensively studied since the late fifties of the last century ([2, pp. 132–135], [3, pp. 311–319]). They have several applications in the determination of substructures of various algebraical structures (see, e.g., [4,8,9]), in the theory of geometric objects and classification of near-rings and quasialgebras (cf. [1,5,22]) as well as in some problems in meteorology and fluid mechanics [20]. There are also strict connections between (2) and some classes of subgroups of the Lie group L^1_{n+1}. The solutions of (1) and (2) have been studied under various regularity assumptions, e.g., in [4,6–10,12,19] and [23]. For more details concerning (1) and (2), its applications and further generalizations we refer to a survey paper [11].

E-mail address: chudziak@univ.rzeszow.pl.

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Recently the stability problem for (1) and (2) has been considered in [13–16]. It has been proved in [15] that for every $n \in \mathbb{N}$ and $\lambda \in K \setminus \{0\}$, Eq. (2) is superstable in the class of functions $f : X \to K$ continuous at 0 on rays, i.e., every such function satisfying the inequality

$$\left|f\left(x + f(x)^n y\right) - \lambda f(x)f(y)\right| \leqslant \varepsilon \quad \text{for } x, y \in X, \tag{3}$$

where $\varepsilon \geqslant 0$, either is bounded or satisfies (2). It is known (see, e.g., [17,18]) that the superstability is caused by the fact that posing a problem (3), one mix two operations. Namely, on the right-hand side of (2) we have a product, but in (3) the distance between the sides of (2) is measured by the difference. As it has been proposed in [17], it is more natural to measure the distance between 1 and the quotients of the sides of (2). This approach leads to a system of two conditional functional inequalities

$$\left|\frac{f(x + f(x)^n y)}{\lambda f(x)f(y)} - 1\right| \leqslant \varepsilon \tag{4}$$

for $x, y \in X$ such that $f(x)f(y) \neq 0$;

$$\left|\frac{\lambda f(x)f(y)}{f(x + f(x)^n y)} - 1\right| \leqslant \varepsilon \tag{5}$$

for $x, y \in X$ such that $f(x + yf(x)) \neq 0$. In the case where $X = K = \mathbb{R}$ and $n = \lambda = 1$, this system has been already considered in [13].

It is the aim of this paper to study the solutions of the system (4)–(5) in a class of functions satisfying a weak regularity assumption. A crucial role in the proof of our main result plays a rather surprising fact that in a wide class of functions $f : X \to K$, Eq. (1) is equivalent to the condition

$$f\left(x + f(x)y\right) = 0 \quad \text{if and only if} \quad f(x)f(y) = 0 \tag{6}$$

for $x, y \in X$. This result seems to be of independent interest.

In what follows X is a vector space over a field K of real or complex numbers and $S(x, r)$, $B(x, r)$ and $\overline{B}(x, r)$ denote, respectively, a sphere, an open ball and a closed ball (in K), with a center at $x \in K$ and a radius $r > 0$. Furthermore, for a given function $f : X \to K$, we put $F_f := \{x \in X\colon\ f(x) = 0\}$. Let us also recall (see, e.g., [21, p. 596]) that if $A \subset X$, then $a \in A$ is called an algebraically interior point of A, provided, for every $x \in X \setminus \{0\}$, there is $r_x > 0$ such that $a + B(0, r_x)x = \{a + bx\colon\ b \in B(0, r_x)\} \subset A$. In the sequel, by $\operatorname{alg\,int} A$ we denote the set of all algebraically interior points of A.

2. Equivalence of (1) and (6)

The following result establishes the equivalence of (1) and (6).

Theorem 1. *Assume that $f : X \to K$ is such that $F_f \neq \emptyset$ and $\operatorname{alg\,int}(X \setminus F_f) \neq \emptyset$. Then f satisfies (6) if and only if f is a solution of (1).*

Proof. Clearly, (1) implies (6). So, assume that f satisfies (6). For every $x \in X \setminus F_f$, define a function $\phi_x : X \to X$ by

$$\phi_x(y) = x + f(x)y \quad \text{for } y \in X.$$

Then, for every $x \in X \setminus F_f$, ϕ_x is bijective and, by (6), the set F_f is strongly ϕ_x-invariant, i.e.,

$$\phi_x(F_f) = F_f. \tag{7}$$

First we prove that $0 \in \operatorname{alg\,int}(X \setminus F_f)$. Obviously, $0 \in X \setminus F_f$, because otherwise applying (6) with $y = 0$, we get $f \equiv 0$, which contradicts the assumptions. Let $x \in X \setminus \{0\}$. Then, taking $a \in \operatorname{alg\,int}(X \setminus F_f)$, we have $a + B(0, r_x)x \subset X \setminus F_f$ for some $r_x > 0$. Hence in view of (7), we obtain

$$B\left(0, \frac{r_x}{|f(a)|}\right)x = \phi_a^{-1}\left(a + B(0, r_x)x\right) \subset X \setminus F_f.$$

This means that $0 \in \operatorname{alg\,int}(X \setminus F_f)$. Next, fix $x, y \in X \setminus F_f$ and put

$$F(x, y) := \frac{f(x + f(x)y)}{f(x)f(y)}.$$

Let a function $\psi : X \to X$ be given by

$$\psi(v) = F(x, y)v \quad \text{for } v \in X.$$

Note that $\psi = \phi_y^{-1} \circ \phi_x^{-1} \circ \phi_{\phi_x(y)}$, so applying (7), we get that a set F_f is strongly ψ-invariant. Let $z_0 \in F_f$. Since $0 \in \operatorname{alg\,int}(X \setminus F_f)$, there is $r > 0$ such that

$$B(0, r)z_0 \subset X \setminus F_f. \tag{8}$$

Furthermore, as F_f is a strongly ψ-invariant set, we have

$$\psi^n(z_0) = F(x, y)^n z_0 \in F_f \quad \text{for } n \in \mathbb{Z}.$$

Hence, by (8), we get

$$\bigl|F(x, y)\bigr| = 1. \tag{9}$$

Let $K_0 = \{k \in K\colon\ kz_0 \in F_f\}$. Then, in view of (7) and (8), for every $k \in K \setminus K_0$, we obtain

$$B\bigl(k, \bigl|f(kz_0)\bigr|r\bigr)z_0 = \bigl(k + f(kz_0)B(0, r)\bigr)z_0 = \phi_{kz_0}\bigl(B(0, r)z_0\bigr) \subset X \setminus F_f,$$

whence

$$B\bigl(k, \bigl|f(kz_0)\bigr|r\bigr) \subset K \setminus K_0.$$

Therefore K_0 is a nonempty (because $1 \in K_0$) closed set and $0 \notin K_0$. So there exists $k_0 \in K_0$ such that

$$|k_0| = \inf\bigl\{|k|\colon\ k \in K_0\bigr\}. \tag{10}$$

We show that

$$f(\alpha k_0 z_0) = 1 - \alpha \quad \text{for } \alpha \in (0, 1). \tag{11}$$

Fix $\alpha \in (0, 1)$. Since $\alpha k_0 \in K \setminus K_0$ and $B(0, |k_0|) \subset K \setminus K_0$, making use of (7), we get

$$B\bigl(\alpha k_0, \bigl|f(\alpha k_0 z_0)k_0\bigr|\bigr)z_0 = \alpha k_0 z_0 + f(\alpha k_0 z_0)B\bigl(0, |k_0|\bigr)z_0 = \phi_{\alpha k_0 z_0}\bigl(B\bigl(0, |k_0|\bigr)z_0\bigr) \subset X \setminus F_f.$$

Hence

$$B\bigl(\alpha k_0, \bigl|f(\alpha k_0 z_0)k_0\bigr|\bigr) \subset K \setminus K_0.$$

Moreover $k_0 \in K_0$, which implies that

$$\bigl|f(\alpha k_0 z_0)k_0\bigr| \leqslant |\alpha k_0 - k_0| = (1 - \alpha)|k_0|$$

and so

$$\bigl|f(\alpha k_0 z_0)\bigr| \leqslant 1 - \alpha. \tag{12}$$

On the other hand, in view of (7), we get

$$F_f \ni \phi_{\alpha k_0 z_0}(k_0 z_0) = \bigl(\alpha + f(\alpha k_0 z_0)\bigr)k_0 z_0.$$

Thus

$$\bigl(\alpha + f(\alpha k_0 z_0)\bigr)k_0 \in K_0,$$

so by (10)

$$\bigl|\alpha + f(\alpha k_0 z_0)\bigr| \geqslant 1.$$

Consequently, using (12), we obtain

$$f(\alpha k_0 z_0) \in \overline{B}(0, 1-\alpha) \setminus B(-\alpha, 1) = \{1-\alpha\}.$$

In this way we have proved (11). Now, as F_f is ψ-strongly invariant, applying (7) and (11), for every $\alpha \in (0,1)$, we get

$$\begin{aligned} F_f &\ni (\phi_{\alpha k_0 z_0} \circ \psi)(k_0 z_0) = \phi_{\alpha k_0 z_0}\big(F(x,y) k_0 z_0\big) \\ &= \alpha k_0 z_0 + f(\alpha k_0 z_0) F(x,y) k_0 z_0 = \big(\alpha + (1-\alpha)F(x,y)\big) k_0 z_0. \end{aligned}$$

Thus

$$\big(\alpha + (1-\alpha)F(x,y)\big) k_0 \in K_0 \quad \text{for } \alpha \in (0,1),$$

whence by (9) and (10), we obtain

$$1 \leqslant \big|\alpha + (1-\alpha)F(x,y)\big| \leqslant \alpha + (1-\alpha)\big|F(x,y)\big| = 1 \quad \text{for } \alpha \in (0,1).$$

Therefore

$$\big|\alpha + (1-\alpha)F(x,y)\big| = 1 \quad \text{for } \alpha \in (0,1).$$

This means that a segment joining 1 and $F(x,y)$ is contained in $S(0,1)$ (note that by (9), $F(x,y) \in S(0,1)$). As K is a strictly convex space, this implies that $F(x,y) = 1$ and so

$$f\big(x + f(x)y\big) = f(x)f(y). \tag{13}$$

Since in the case where $x \in F_f$ or $y \in F_f$, (13) results from (6), the proof is completed. □

Remark 1. It is clear that in Theorem 1, the assumption $F_f \neq \emptyset$ is essential. To see that the assumption $\operatorname{alg\,int}(X \setminus F_f) \neq \emptyset$ is also essential it is enough to take a function $f : X \to K$ of the form

$$f(x) = \begin{cases} k_0 & \text{if } x = 0, \\ 0 & \text{otherwise}, \end{cases}$$

where $k_0 \in K \setminus \{0,1\}$ is fixed. Then $F_f \neq \emptyset$, $\operatorname{alg\,int}(X \setminus F_f) = \emptyset$ and, as a straightforward calculation shows, f satisfies (6), but it is not a solution of (1).

3. Stability

In the sequel, given $n \in \mathbb{N}$ and $\lambda \in K \setminus \{0\}$, we set $E_n(K) := \{k \in K\colon\ k^n = 1\}$ and

$$\Delta(\lambda, n, K) := \min\big\{|\lambda - e_n|\colon\ e_n \in E_n(K) \setminus \{\lambda\}\big\} \max\left\{1, \frac{1}{|\lambda|}\right\}. \tag{14}$$

We begin this section with the following remark concerning the solutions of the system (4)–(5).

Remark 2. Assume that $n \in \mathbb{N}$, $\lambda \in K \setminus \{0\}$ and $\varepsilon \in [0,1)$. If $f : X \to K$ satisfies the system (4)–(5) and $F_f = \emptyset$, then

$$f(X) \subset E_n(K) \cup \frac{1}{\lambda}\left[\overline{B}(1,\varepsilon) \cap \frac{1}{1-\varepsilon^2}\overline{B}(1,\varepsilon)\right]. \tag{15}$$

In fact, suppose that $f(x) \notin E_n(K)$ for some $x \in X$. Then, taking $y = \frac{x}{1-f(x)^n}$ and using (4) and (5), we obtain

$$\left|\frac{1}{\lambda f(x)} - 1\right| \leqslant \varepsilon \tag{16}$$

and

$$\big|\lambda f(x) - 1\big| \leqslant \varepsilon, \tag{17}$$

respectively. Let $i : K \setminus \{0\} \to K \setminus \{0\}$ be the inversion, i.e., $i(k) = \frac{1}{k}$ for $k \in K \setminus \{0\}$. Then i maps the ball $\overline{B}(1, \varepsilon)$ onto the ball $\overline{B}(\frac{1}{1-\varepsilon^2}, \frac{\varepsilon}{1-\varepsilon^2})$. Since, by (16), $i(\lambda f(x)) \in \overline{B}(1, \varepsilon)$, this implies that

$$\lambda f(x) = (i \circ i)\big(\lambda f(x)\big) \in i\big(\overline{B}(1, \varepsilon)\big) = \overline{B}\left(\frac{1}{1-\varepsilon^2}, \frac{\varepsilon}{1-\varepsilon^2}\right) = \frac{1}{1-\varepsilon^2}\overline{B}(1, \varepsilon).$$

Thus, making use of (17), we obtain (15).

The next theorem is a main result of the paper.

Theorem 2. *Let $n \in \mathbb{N}$, $\lambda \in K \setminus \{0\}$ and $\varepsilon \in [0, \min\{1, \Delta(\lambda, n, K)\})$. Assume that a function $f : X \to K$ satisfies the system* (4)–(5) *and* $\operatorname{alg\,int}(X \setminus F_f) \neq \emptyset$. *Then either* (15) *is valid, or f is a solution of* (2).

Proof. According to Remark 2, it suffices to consider the case where $F_f \neq \emptyset$. We show that in this case, (2) holds. To this end fix $x, y \in X$. Since $\varepsilon < 1$, from (4) and (5) we derive that

$$f\big(x + f(x)^n y\big) = 0 \quad \text{if and only if} \quad f(x)f(y) = 0. \tag{18}$$

Thus, if $x \in F_f$ or $y \in F_f$, then (2) is valid. So, assume that $x, y \in X \setminus F_f$. Note that from (18) it follows that the function $\bar{f} := (f)^n$ satisfies (6). Furthermore, we have $F_{\bar{f}} = F_f$, whence $F_{\bar{f}} \neq \emptyset$ and $\operatorname{alg\,int}(X \setminus F_{\bar{f}}) \neq \emptyset$. Therefore, applying Theorem 1, we conclude that $\bar{f}$ is a solution of (1). Hence

$$\left(\frac{f(x + f(x)^n y)}{f(x)f(y)}\right)^n = \frac{\bar{f}(x + \bar{f}(x)y)}{\bar{f}(x)\bar{f}(y)} = 1$$

and so

$$\frac{f(x + f(x)^n y)}{f(x)f(y)} \in E_n(K). \tag{19}$$

On the other hand, in view of (4) and (5), we have

$$\big|f\big(x + f(x)^n y\big) - \lambda f(x)f(y)\big| \leqslant \varepsilon \min\big\{\big|f\big(x + f(x)^n y\big)\big|, \big|\lambda f(x)f(y)\big|\big\}.$$

Thus, applying (14) and (19), we obtain

$$\begin{aligned}\left|\frac{f(x + f(x)^n y)}{f(x)f(y)} - \lambda\right| &\leqslant \varepsilon \min\left\{\left|\frac{f(x + f(x)^n y)}{f(x)f(y)}\right|, |\lambda|\right\} = \varepsilon \min\big\{1, |\lambda|\big\} \\ &< \Delta(\lambda, n, K)\min\big\{1, |\lambda|\big\} = \min\big\{|\lambda - e_n|\colon\ e_n \in E_n(K) \setminus \{\lambda\}\big\}.\end{aligned}$$

Consequently

$$\frac{f(x + f(x)^n y)}{f(x)f(y)} \notin E_n(K) \setminus \{\lambda\},$$

which in view of (19) gives

$$\frac{f(x + f(x)^n y)}{f(x)f(y)} = \lambda.$$

Therefore f satisfies (2). □

The next example shows that the constant $\Delta(\lambda, n, K)$ is optimal, in a sense that if $\Delta(\lambda, n, K) < 1$, then for every $\varepsilon \in [\Delta(\lambda, n, K), 1)$ there exists a function $f : X \to K$ with $\operatorname{alg\,int}(X \setminus F_f) \neq \emptyset$, satisfying the system (4)–(5) and such that neither f is bounded nor f is a solution of (2).

Example 1. Assume that $n \in \mathbb{N}$ and $\lambda \in K \setminus \{0\}$ are such that $\Delta(\lambda, n, K) < 1$. Let $\varepsilon \in [\Delta(\lambda, n, K), 1)$. Then, by (14), there exists $p \in E_n(K) \setminus \{\lambda\}$ with

$$|\lambda - p| \max\left\{1, \frac{1}{|\lambda|}\right\} \leqslant \varepsilon. \tag{20}$$

Let $L : X \to \mathbb{R}$ be an arbitrary nontrivial $\mathbb{R}$-linear functional. Define a function $f : X \to K$ by

$$f(x) = \frac{1}{p}\bigl(\max\bigl\{L(x) + 1, 0\bigr\}\bigr)^{\frac{1}{n}} \quad \text{for } x \in X.$$

First, we show that $0 \in \operatorname{alg\,int}(X \setminus F_f)$. To this end take $x \in X \setminus \{0\}$ and note that a function $L_x : K \to \mathbb{R}$ given by $L_x(t) = L(tx)$ for $t \in K$ is $\mathbb{R}$-linear and so continuous. Thus, a function $f_x : K \to \mathbb{R}$ of the form $f_x(t) = \frac{1}{p}(\max\{L_x(t) + 1, 0\})^{\frac{1}{n}}$ for $t \in K$ is also continuous. Furthermore, as $f_x(0) = f(0) = \frac{1}{p} \neq 0$, there is $r_x > 0$ such that $0 \notin f_x(B(0, r_x)) = f(B(0, r_x)x)$, whence $B(0, r_x)x \subset X \setminus F_f$. In this way we have proved that $0 \in \operatorname{alg\,int}(X \setminus F_f) \neq \emptyset$. Next, an easy computation gives

$$f\bigl(x + f(x)^n y\bigr) = pf(x)f(y) \quad \text{for } x, y \in X, \tag{21}$$

so f is not a solution of (2). Moreover, f is unbounded and, making use of (20) and (21), we get

$$\left|\frac{f(x + f(x)^n y)}{\lambda f(x)f(y)} - 1\right| = \left|\frac{p}{\lambda} - 1\right| \leqslant \varepsilon$$

for $x, y \in X$ such that $f(x)f(y) \neq 0$; and

$$\left|\frac{\lambda f(x)f(y)}{f(x + f(x)^n y)} - 1\right| = \left|\frac{\lambda}{p} - 1\right| = \frac{|\lambda - p|}{|p|} = |\lambda - p| \leqslant \varepsilon$$

for $x, y \in X$ such that $f(x + f(x)^n y) \neq 0$. Therefore, f satisfies the system (4)–(5).

Finally, we will prove the following result describing the solutions of (2).

Theorem 3. *Assume that $n \in \mathbb{N}$, $\lambda \in K \setminus \{0\}$ and $\varepsilon \in [0, \min\{1, \Delta(\lambda, n, K)\})$. Let $f : X \to K$ be a solution of* (4)–(5) *such that* $\operatorname{alg\,int}(X \setminus F_f) \neq \emptyset$.

(i) *If $\lambda \notin E_n(K)$, then f satisfies* (15).
(ii) *If $\lambda \in E_n(K)$, then either f satisfies* (15)*, or one of the following conditions holds.*
 (A) *There exists a nontrivial $\mathbb{R}$-linear functional $L : X \to \mathbb{R}$ such that*
  1° *in the case where n is an odd number,*

$$f(x) = \frac{1}{\lambda}\bigl(L(x) + 1\bigr)^{\frac{1}{n}} \quad \textit{for } x \in X,$$

  or

$$f(x) = \frac{1}{\lambda}\bigl(\max\bigl\{L(x) + 1, 0\bigr\}\bigr)^{\frac{1}{n}} \quad \textit{for } x \in X; \tag{22}$$

  2° *in the case where n is even, f is of the form* (22);
 (B) $n = 1$ (*and so* $\lambda = 1$) *and there exists a nontrivial $\mathbb{C}$-linear functional $L : X \to \mathbb{C}$ such that*

$$f(x) = L(x) + 1 \quad \textit{for } x \in X.$$

Proof. Assume that (15) does not hold. Then, according to Remark 2, $F_f \neq \emptyset$. Hence, by Theorem 2, f satisfies (2). On the other hand, since $\varepsilon < 1$, from (4) and (5) it follows (18). Therefore, according to Theorem 1, we get

$$f\bigl(x + f(x)^n y\bigr)^n = f(x)^n f(y)^n \quad \text{for } x, y \in X,$$

which in view of (2) means that $\lambda \in E_n(K)$. Now, taking $\bar{f} := \lambda f$, we obtain by (2)

$$\bar{f}\bigl(x + \bar{f}(x)^n y\bigr) = \bar{f}(x)\bar{f}(y) \quad \text{for } x, y \in X.$$

Furthermore, as $\operatorname{alg\,int}(X \setminus F_{\bar{f}}) = \operatorname{alg\,int}(X \setminus F_f) \neq \emptyset$, applying [9, Theorem 3], we get the assertion. □

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